

**ABSTRACT**

*In this paper, we examine the use of geometric versus arithmetic assumptions when constructing return expectations for use in mean-variance optimization and the generation of expected portfolio returns. In a mean-variance framework, we observe that an expected return derived from an arithmetic mean tends to overstate the long-term realized returns, while an expected return derived from a geometric mean tends to understate the realized return. Accordingly, we recommend using a blended approach that we call a modified geometric mean.*

**BACKGROUND**

Return assumptions are important because they drive asset allocation, spending, and contribution decisions. There is, however, a surprising lack of agreement among practitioners on what definition of returns to use when calculating expected returns. Geometric and arithmetic returns, both defined below, can be very different. Yet both types of returns are used to decide critical issues, such as appropriate investment policy weights.

The geometric average return (or “geometric mean”) is the rate of return that an investment must earn in each period, over a number of periods (e.g., months or years), to arrive at some future value. The geometric mean answers the question: At what rate must an investment pool grow to equal a predetermined value at some future date, assuming periodic compounding? The arithmetic average return (“arithmetic mean”), on the other hand, is indifferent to compounding and amounts to a simple average. The two are only the same when returns are perfectly steady. If returns are volatile, the arithmetic mean will exceed the geometric mean. Moreover, the greater the volatility, the greater the difference will be between the two means.

Stocks and bonds fluctuate in value – that is, they are volatile. For this reason, using geometric means, a portfolio composed of 60% US stocks (expected return of 8.8%) and 40% core bonds (expected return of 3.2%) would have an expected return of 6.6% per annum over twenty years.<sup>1</sup> However, using arithmetic means, this same portfolio would have an expected return of 7.7%.

The geometric mean better captures the experience of an investor during periods of volatility. This is illustrated by the example in the following table. In this example, a dollar invested in an asset loses 25% in Period 1 and gains 25% in Period 2. The arithmetic mean yields a return of zero, yet the investor has lost money (6.25% of the original investment). The geometric mean captures this loss (expressing it as an annual average), while the arithmetic mean does not. Hence, the geometric mean tends to better represent an investor’s actual experience. However, this does not make the geometric mean the “right” definition of return to use in all circumstances.

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<sup>1</sup> Return expectations are from Meketa Investment Group’s 2012 Asset Study.

Return in Period 1	-25%
Return in Period 2	25%
Cumulative Return	-6.3%
Arithmetic Mean	0.0%
Geometric Mean	-3.2%

### WHICH SHOULD WE USE?

The example above highlights how the same asset returns can be described very differently using the two approaches. The meaningful difference between the two approaches raises the question of whether to use arithmetic or geometric means when trying to define future return expectations for a portfolio. The short answer is that some combination of the two is probably appropriate.

This is because the arithmetic mean has systematically overestimated the future realized return of a portfolio over long time periods, while the geometric mean has underestimated it. Beyond the mathematical difference described above, financial literature has examined two distinct explanations for this discrepancy, but has yet to come to a consensus.

### NON-NORMAL RETURNS

The first explanation has to do with the “noise” of the estimates (i.e., how precise are the return expectations).<sup>2</sup> The underlying math is fairly complex, but it boils down to the effect of returns in financial markets not being normally distributed (i.e., the path of possible outcomes is not smooth).

Because investors lack clairvoyance when making projections of future asset returns, there is likely to be a difference between the ex-ante projected returns and the ex-post realized returns. This difference, known as an estimation error, may be either positive or negative.

Positive estimation errors have a greater impact than negative estimation errors of an equal magnitude because returns are not normally distributed, but skewed to the right (i.e., higher returns). For short investment horizons, the effect of skewness is minimal. However, the effect of skewness over long horizons (e.g., twenty years) can be quite significant.

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<sup>2</sup> See Jacquier, Kane & Marcus, “Geometric or Arithmetic Mean: a Reconsideration,” *Financial Analyst Journal*, 2003.

## REBALANCING

The second explanation, which is more intuitive, is related to what may be referred to as a rebalancing effect. The rebalancing effect is also referred to as a “diversification return” in the academic literature,<sup>3</sup> and it has been cited in and among other asset classes (e.g., commodities<sup>4</sup>).

The following example illustrates this concept. Assume that an investor had perfect foresight in 1926 and knew that over the next 85 years US stocks would return 9.9% and bonds would return 5.4% per annum (geometrically compounded). Standard MVO calculations will take a weighted average of these geometric means to calculate the expected return for the portfolio.

1926-2011	S&P 500 (%)	IA SSBI U.S. IT Gov't (%)	Predicted 60/40 (%)	Actual 60/40 (%)
Arithmetic Mean	11.9%	5.5%	9.3%	9.3%
Geometric Mean	9.9%	5.4%	8.1%	8.6%

By taking a weighted average for a 60/40 portfolio, the MVO software would have produced an expected return of 8.1% ( $0.6 \times 9.9\% + 0.4 \times 5.4\%$ ). Yet, if the investor had actually implemented the 60/40 portfolio in 1926 and rebalanced annually, the actual return would have been 8.6%. And, even if the investor had changed the rebalancing time frame (e.g., to quarterly or monthly), the result would have been roughly the same.

The difference between the two returns (8.6% actual and 8.1% predicted) may be referred to as a rebalancing “bonus.” This bonus (i.e., the diversification return) results from the variance reduction that occurs when a portfolio is regularly rebalanced. On a more basic (and somewhat oversimplified) level, the underlying idea is that the bonus results from buying low and selling high on a relative basis.

The magnitude of the diversification return is determined by several factors. Specifically, the diversification return is highest when 1) the average variance of the securities in a portfolio is high, 2) the average correlation of the securities in the portfolio is low, and 3) the number of securities (or asset classes) in the portfolio is high.

## SEEKING COMPROMISE

While the available literature and historical evidence argue for using a number that is higher than the geometric mean and lower than the arithmetic mean, they have failed to produce an industry standard method for arriving at this figure. One article suggests that the amount of weight put on one versus the other should depend on the time horizon for which returns are

<sup>3</sup> See Bernstein & Wilkinson, “Diversification, Rebalancing, and the Geometric Mean Frontier,” 1997.

<sup>4</sup> See, for example, Erb & Harvey, “The Tactical and Strategic Value of Commodity Futures,” 2006.

being forecasted.<sup>5</sup> It suggests that the longer the time horizon, the closer the number should be to the geometric mean, and vice versa.

Because there is no single correct formula or set of assumptions to use, we have adopted an approach that combines the two numbers. Because Meketa Investment Group's Asset Study makes twenty-year projections (i.e., a fairly long-term time horizon), we choose to weight the geometric mean more heavily than the arithmetic mean. Specifically, we are putting a 75% weight on the geometric mean and a 25% weight on the arithmetic mean.

We conducted multiple tests of this methodology, including the 60-40 stock-bond example described previously. As illustrated in the following table, this "modified geometric mean" would have resulted in an 8.4% expected return, which was much closer to the actual return.

Since 1926	S&P 500 (%)	IA SSBI U.S. IT Gov't (%)	Predicted 60/40 (%)	Actual 60/40 (%)
Arithmetic Mean	11.9	5.5	9.3	9.3
Geometric Mean	9.9	5.4	8.1	8.6
Modified E(R)	10.4	5.4	8.4	8.6

It is important to note the distinction between inputs and outputs of the MVO model. "Inputs," or return assumptions, are derived in a variety of ways, but most practitioners interpret their long-term return assumptions as "single period" returns, or arithmetically derived (i.e., the returns will be lower once multi-period compounding is taken into account). If inputs are arithmetic, then the multi-period output from the MVO calculator will be geometric. Our adjustment increases the inputs, which results in a change to the outputs.

#### CAVEATS

When using this approach, an investor should be aware of potential drawbacks. First, this approach assumes a long-term time-horizon. If the forecast is being made for shorter horizons (e.g., five years), then a modification to the inputs may be appropriate.

In addition, the modified geometric mean will likely overstate the expected return for portfolios concentrated in a volatile asset, but may understate the expected return for well-diversified portfolios that are regularly rebalanced. Moreover, the modified geometric mean will overstate the expected return for a portfolio that is not rebalanced.

<sup>5</sup> See Jacquier, Kane & Marcus, "Geometric or Arithmetic Mean: a Reconsideration," Financial Analyst Journal, 2003.

### CONCLUSION

The return assumption used by investors is important to many of the decisions that they make. Hence, we seek to produce the most appropriate return assumption. We observe that arithmetic means tend to overstate long-term returns, while geometric means tend to understate them. Both the available literature and historical evidence argue for using a number that is higher than the geometric mean but lower than the arithmetic mean.

Because there is no single correct formula or set of assumptions to use, we have adopted an approach that blends the two means. Specifically, we have constructed a modified geometric mean that places a 75% weight on the geometric mean and a 25% weight on the arithmetic mean. Our analysis suggests that the modified geometric mean generates expected returns that are more consistent with the actual returns experienced by rebalanced portfolios. Hence, we believe the modified geometric mean is superior to either the geometric or arithmetic mean for the purpose of projecting portfolio returns.